An Invariance Principle for Fractional Brownian Sheets

Yizao Wang*

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Abstract

We establish a central limit theorem for partial sums of stationary linear random fields with dependent innovations, and an invariance principle for anisotropic fractional Brownian sheets. Our result is a generalization of the invariance principle for fractional Brownian motions by Dedecker et al. [9] to high dimensions. A key ingredient of their argument, the martingale approximation, is replaced by an m-approximation argument. An important tool of our approach is a moment inequality for stationary random fields recently established by El Machkouri et al. [16].

Keywords: m-dependence; central limit theorem; invariance principle; fractional Brownian sheet; fractionally integrated random field.
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1 Introduction

Consider a stationary linear random field $\{\xi_i\}_{i\in\mathbb{Z}^d}$ with stationary mean-zero innovations $\{X_i\}_{i\in\mathbb{Z}^d}$:

$$\xi_j = \sum_{i\in\mathbb{Z}^d} a_{j-i}X_i \,, j\in\mathbb{Z}^d \,, \tag{1}$$

where $\{a_i\}_{i\in\mathbb{Z}^d}$ are a collection of real numbers such that $\sum_{i\in\mathbb{Z}^d} a_i^2 < \infty$. In particular, we are interested in the partial sum $S_n := \sum_{i\in\{1,\dots,n\}^d} \xi_i$. Set $b_{n,j} = \sum_{i\in\{1,\dots,n\}^d} a_{i-j}$ and $b_n = (\sum_{j\in\mathbb{Z}^d} b_{n,j}^2)^{1/2}$. Then $S_n = \sum_{j\in\mathbb{Z}^d} b_{n,j}X_j$.

We establish sufficient conditions for the following two problems:

(i) When do we have a *central limit theorem*

$$\frac{S_n}{b_n} \equiv \frac{\sum_{j\in\mathbb{Z}^d} b_{n,j}X_j}{b_n} \Rightarrow \mathcal{N}(0,\sigma^2) \quad ?$$

*Department of Statistics, The University of Michigan, 439 W. Hall, 1085 S. University, Ann Arbor, MI 48109–1107. U.S.A. *E-mail:* `yizwang@umich.edu`. Fax: +1 7347634676.

(ii) When do we have an *invariance principle*

$$\left\{\frac{S_{\lfloor nt\rfloor}}{b_n}\right\}_{t\in[0,1]^d} \Rightarrow \{\mathbb{G}_t\}_{t\in[0,1]^d}$$

and what is the limiting process $\mathbb{G}$?

These two problems have a long history. In the one-dimensional case ($d = 1$), they have been extensively investigated and many results are known. For example, when $\{X_i\}_{i\in\mathbb{Z}}$ are independent and identically distributed (i.i.d.), Davydov in the seminal work [6] established an invariance principle for fractional Brownian motion (we will recall the definition below). See also Konstantopoulos and Sakhanenko [24] for a sharper condition. When $\{X_i\}_{i\in\mathbb{Z}}$ are stationary, the central limit theorem have been considered by for example Peligrad and Utev [35, 36], Wu and Woodroofe [47], Merlevède and Peligrad [30], among others. The invariance principle has been investigated for *fractionally integrated processes*, an important class of our model (1) (see e.g. Wu and Shao [46] and references therein for more results from the point of view of fractionally integrated processes), and recently by Dedecker et al. [9] in the general setting.

However, very few of the corresponding results in high-dimensional case ($d \geq 2$) are known, with the notable exceptions of Surgailis [38] and Lavancier [28]. Surgailis [38] established general convergence results for (functionals of) linear random fields with independent innovations to *self-similar* random fields. Lavancier [28] investigated linear random fields with dependent innovations via a spectral convergence theorem (see also Lang et Soulier [25]). In [28], the limiting objects are described in terms of linear mappings, and fractional Brownian sheets appear explicitly in the limit only when $\{X_i\}_{i\in\mathbb{Z}^d}$ are i.i.d. random variables with zero mean and finite variance.

Recall that, for a given vector $H = (H_1,\dots,H_d) \in (0,1)^d$, a (real-valued) *fractional Brownian sheet* $\mathbb{B}^H = \{\mathbb{B}^H_t\}_{t\in[0,1]^d}$ with *Hurst index* H is a real-valued mean-zero Gaussian random fields with covariance function given by

$$\mathbb{E}(\mathbb{B}^H_s\mathbb{B}^H_t) = \prod_{q=1}^d \frac{1}{2}\Big(s_q^{2H_q} + t_q^{2H_q} - |s_q - t_q|^{2H_q}\Big), s,t\in[0,1]^d\,. \qquad (2)$$

When $d = 1$, $\mathbb{B}^H$ is the fractional Brownian motion with Hurst index H_1. Fractional Brownian sheets play an important role in modeling anisotropic random fields with *long-range dependence* (see e.g. Doukhan et al. [14] and Lavancier [27]). They also arise in the study of stochastic partial differential equations (e.g. Hu et al. [23] and Øksendal and Zhang [34]). The investigation of their sample path properties is another active research area (e.g. Xiao [48]).

When $H_q = 1/2$ for all $q = 1, \ldots, d$ in (2), the fractional Brownian sheet is just the (multiparameter) Brownian sheet, whence the former is a natural extension of the latter. Central limit theorems for stationary random fields and weak convergence to Brownian sheets have been considered by several authors, including Basu and Dorea [1], Bolthausen [3], Nahapetian and Petrosian [33], Nahapetian [32], Poghosyan and Roelly [37], Dedecker [7, 8], El Machkouri [15], Cheng and Ho [5], Wang and Woodroofe [41] and El Machkouri et al. [16], among others. In particular, people have recently found that in order to establish a central limit theorem for stationary random fields, a convenient way is to approximate the dependent random variables by m-dependent ones: the central limit theorems for m-dependent random variables have been known since Hoeffding and Robbins [22], and it remains to show that the difference becomes arbitrarily small as m increases. This *m-approximation* method has been successfully applied to stationary random fields by Wang and Woodroofe [41] and El Machkouri et al. [16], under different conditions measuring the dependence of random fields. At the same time, the m-approximation method has also been successful in one dimension. For example, recently Liu and Lin [29] applied this method to establish a strong approximation of stationary sequences by Brownian motions with optimal rates.

This work develops an invariance principle for linear random fields with *dependent innovations*. Our result, as an invariance principle for fractional Brownian sheets, improves the ones by Surgailis [38] and Lavancier [28], as they both required innovations to be i.i.d. At the same time, their results are more general in the sense that they include weak convergence results with other limiting objects, while the ours does not.

Our result can be seen as an extension of Dedecker et al. [9] to high dimensions. The main difference is that we replace their martingale approximation method, which seems difficult to be generalized to high dimensions, by the m-approximation method. Besides, to characterize the weak dependence of innovations, we apply the *physical dependence measure* introduced by Wu [43] and extended to random fields by El Machkouri et al. [16]. In particular, El Machkouri et al. [16] proved a moment inequality of weighted partial sums of $\{X_i\}_{i\in\mathbb{Z}^d}$, which is of significant importance in the analysis of random fields by m-approximation.

Another crucial assumption for our results is a product structure for the coefficients:

$$a_i = \prod_{q=1}^{d} a_{i_q}(q)\,, i \in \mathbb{Z}^d\,,$$

and $\{a_{i_q}(q)\}_{i_q\in\mathbb{Z}}$ are square-summable real numbers for each $q = 1, \ldots, d$. In particular, the product structure allows us to extend the idea of *coefficient-averaging* by Peligrad and Utev [36] to high dimensions. It also plays an important role in the analysis of asymptotic covariance structure.

The product structure of the coefficients is an reasonable assumption and it was assumed in [28] to have an invariance principle for fractional Brownian sheets. To see that it is not a restrictive assumption, recall the product form of the covariance formula (2), and the fact that fractional Brownian sheets have the following stochastic integral representation with a product kernel $\prod_{q=1}^d g_{H_q}$:

$$\mathbb{B}^H(t) = \frac{1}{\kappa_H}\int_{-\infty}^{t_1}\cdots\int_{-\infty}^{t_d}\prod_{q=1}^d g_{H_q}(t_q,s_q)\mathbb{W}(\mathrm{d}s)\,,$$

where $\mathbb{W} = \{\mathbb{W}_s\}_{s\in\mathbb{R}^d}$ is a standard Brownian sheet, $g_{H_q}(t_q,s_q) = ((t_q - s_q)_+)^{H_q-1/2} - ((-s_q)_+)^{H_q-1/2}$ with $r_+ = \max(r,0)$ and κ_H is a normalizing constant (see e.g. [48]).

At last, we point out that a more general problem is to consider partial sums of $\{K(\xi_i)\}_{i\in\mathbb{Z}^d}$ for some function $K:\mathbb{R}\to\mathbb{R}$ (we address the case $K(x) = x$), including the important case of empirical processes. A closely related problem is to consider the asymptotic properties of partial sums of $\{K(Z_i)\}_{i\in\mathbb{Z}^d}$ where $\{Z_i\}_{i\in\mathbb{Z}^d}$ is a Gaussian random field. Such results have significant impacts in statistics theory. These problems again have been extensively investigated in one-dimensional case (see e.g. Taqqu [39], Giraitis and Surgailis [18, 19], Ho and Hsing [20, 21], Wu [42, 44], to mention a few), while high-dimensional cases have been much less considered (see e.g. Dobrushin [11] and Dobrushin and Major [12], Surgailis [38], Doukhan et al. [13] and Lavancier [27]). To extend our results to general functionals of linear random fields is beyond the scope of this paper, but worth investigating in the future.

The paper is organized as follows. We will provide the background on the physical dependence measure in Section 2. A central limit theorem (Theorem 2) and an invariance principle (Theorem 3) are established in Sections 3, and 4 respectively. Discussions on related works are provided in Section 5.

2 Preliminaries on physical dependence measure

Consider stationary random fields $\{X_i\}_{i\in\mathbb{Z}^d}$ of the following form

$$X_i = g(\epsilon_{i-j} : j\in\mathbb{Z}^d), i\in\mathbb{Z}^d\,, \tag{3}$$

where $g:\mathbb{R}^{\mathbb{Z}^d}\to\mathbb{R}$ is a measurable function and $\{\epsilon_i\}_{i\in\mathbb{Z}^d}$ are i.i.d. random variables. Throughout this paper, we assume that $\mathbb{E}X_0 = 0$.

El Machkouri et al. [16] suggested to measure the dependence of $\{X_i\}_{i\in\mathbb{Z}^d}$ as follows. Let $\epsilon^* = \{\epsilon^*_i\}_{i\in\mathbb{Z}^d}$ be a random field coupled with ϵ, defined by $\epsilon^*_i = \epsilon_i$ for all $i\in\mathbb{Z}^d\setminus\{0\}$ and ϵ^*_0 being a copy of ϵ_0 independent of ϵ.

Set $X_i^* = g(\epsilon_{i-j}^* : j \in \mathbb{Z}^d)$ and define the *physical dependence measure* of $\{X_i\}_{i\in\mathbb{Z}^d}$ by

$$\Delta_p \equiv \Delta_p(X) = \sum_{i\in\mathbb{Z}^d} \|X_i - X_i^*\|_p . \tag{4}$$

El Machkouri et al. derived a central limit theorem and an invariance principle under the condition that $\Delta_p < \infty$ for certain $p \geq 2$. In particular, the following result is useful for our purpose.

Theorem 1 (El Machkouri et al. [16]). *(i) Let $\{a_i\}_{i\in\mathbb{Z}^d}$ be a family of real numbers. Then for any $p \geq 2$,*

$$\Big\| \sum_{i\in\mathbb{Z}^d} a_i X_i \Big\|_p \leq \Big(2p \sum_{i\in\mathbb{Z}^d} a_i^2 \Big)^{1/2} \Delta_p . \tag{5}$$

(ii) $\Delta_2 < \infty$ implies that $\sum_{k\in\mathbb{Z}^d} |\mathbb{E}X_0X_k| < \infty$.

In fact, (5) was proved for $\{a_i\}_{i\in\mathbb{Z}^d}$ having finite non-zero numbers, but the extension is immediate.

Our results require stationary random fields $\{X_i\}_{i\in\mathbb{Z}^d}$ to satisfy $\Delta_p < \infty$ for some $p \geq 2$. El Machkouri et al. [16] provided several such examples.

Example 1. Consider $\{X_i\}_{i\in\mathbb{Z}^d}$ in form of functional of linear random fields:

$$X_i = g\Big(\sum_{j\in\mathbb{Z}^d} \psi_{i-j}\epsilon_j \Big), i \in \mathbb{Z}^d ,$$

where g is a Lipschitz continuous function and the coefficients $\{\psi_i\}_{i\in\mathbb{Z}^d}$ satisfy $\sum_{j\in\mathbb{Z}^d} |\psi_j| < \infty$. If $\epsilon_0 \in L^p$ for $p \geq 2$, then $\Delta_p < \infty$ ([16], Example 1). Note that this class of random fields include linear random fields. Another class of non-linear random fields are the Volterra fields ([16], Example 2).

Remark 1. Moment inequalities play an important role in establishing asymptotic results for random fields. Dedecker [8] also established a similar moment inequality for random fields, under a different condition of weak dependence. In principle, in order to establish asymptotic normality one should expect to control, for finite subset $\Gamma \subset \mathbb{Z}^d$, $\| \sum_{i\in\Gamma} X_i\|_p \leq O(|\Gamma|^{1/2})$ for some $p \geq 2$. Wang and Woodroofe [41] established such an inequality for Γ in form of rectangles, under a different condition of weak dependence.

Remark 2. In the literature of (one-dimensional) stationary sequences, such a condition $\Delta_p < \infty$ on the weak dependence is often referred to as of *projective type*. An advantage of projective-type conditions is that, they often lead to easy-to-verify conditions for the asymptotic normality of various stationary processes arising from statistics and econometrics. For more on the projective-type conditions in one dimension, see for example Wu [43, 45] and Merlevède et al. [31]. For other types of conditions on weak dependence, see for example Bradley [4] and Dedecker et al. [10].

3 A central limit theorem

We first establish a central limit theorem for triangular array in form of

$$S_n \equiv S_n(X,b) := \sum_{j\in\mathbb{Z}^d} b_{n,j} X_j \,, \tag{6}$$

with general *regular* coefficients $\{b_{n,j}\}_{n,j}$ to be defined below. For each $l \in \mathbb{N}$, set rectangle blocks of size l^d by

$$I_k \equiv I_k(l) = \{i \in \mathbb{Z}^d : i_q \in \{lk_q + 1, \ldots, lk_q + l\}, q = 1, \ldots, d\} \subset \mathbb{Z}^d \,,$$

and define

$$c_{n,k} = \frac{1}{l^d} \sum_{j\in I_k} b_{n,j} \,, n \in \mathbb{N}, k \in \mathbb{Z}^d \tag{7}$$

and $c_n = (\sum_{k\in\mathbb{Z}^d} c_{n,k}^2)^{1/2}$. We introduce the following definition in the spirit of the coefficient-averaging idea of Peligrad and Utev [36].

Definition 1. We say the coefficients $\{\{b_{n,j}\}_{j\in\mathbb{Z}^d} : n \in \mathbb{N}\}$ are *regular*, if $b_n^2 \to \infty$ and for each $l \in \mathbb{N}$,

$$\lim_{n\to\infty} \frac{1}{b_n^2} \sum_{k\in\mathbb{Z}^d} \sum_{j\in I_k} (b_{n,j} - c_{n,k})^2 = 0 \tag{8}$$

$$\lim_{n\to\infty} \sum_{k\in\mathbb{Z}^d} \sum_{j\in I_k} |b_{n,j}^2 - c_{n,k}^2| = 0 \tag{9}$$

$$\lim_{n\to\infty} \sup_{j\in\mathbb{Z}^d} \frac{|c_{n,j}|}{c_n} = 0. \tag{10}$$

Theorem 2. *Consider S_n as in* (6) *with some regular coefficients $\{b_{n,j}\}_{n,j}$. If $\Delta_2 < \infty$, then, $\sigma^2 := \sum_{k\in\mathbb{Z}^d} \mathbb{E} X_0 X_k < \infty$ and*

$$\frac{S_n}{b_n} \Rightarrow \mathcal{N}(0, \sigma^2) \,. \tag{11}$$

Theorem 2 leads to an answer to our first question. In particular, return to our problem of linear random fields with $b_{n,j} = \sum_{i\in\{1,\ldots,n\}^d} a_{i-j}$. It suffices to show such $\{b_{n,j}\}_{n,j}$ are regular. Recall that the coefficients $\{a_i\}_{i\in\mathbb{Z}^d}$ are assumed to have the product structure:

$$a_i = \prod_{q=1}^d a_{i_q}(q), \text{ for some } \{a_{i_q}(q)\}_{i_q\in\mathbb{Z}}, q = 1, \ldots, d \,, \tag{12}$$

with $\sum_{j\in\mathbb{Z}^d} a_j^2 < \infty$. This suffices to establish the regularity of $\{b_{n,j}\}_{n,j}$.

Corollary 1. *If $b_{n,j} = \sum_{i\in\{1,\ldots,n\}^d} a_{i-j}$ with $\{a_i\}_{i\in\mathbb{Z}^d}$ having the product form* (12)*, then $\{b_{n,j}\}_{n,j}$ is regular. As a consequence, if in addition $\Delta_2 < \infty$, then $S_n(X,b)/b_n \Rightarrow \mathcal{N}(0,\sigma^2)$ with $\sigma^2 = \sum_{j\in\mathbb{Z}^d} \mathbb{E}(X_0 X_j) < \infty$.*

We first prove Theorem 2 through a series of approximations of S_n in (6). The main tool is Theorem 1.

(i): m-approximation. First, fix m and let $\overline{X} = \{\overline{X}_i\}_{i\in\mathbb{Z}^d}$ (depending on m) denote the stationary random field obtained by

$$\overline{X}_i = \mathbb{E}(X_i \mid \mathcal{F}_i^m)\,,$$

where $\mathcal{F}_i^m = \sigma(\epsilon_{i-j} : j \in \{-\lfloor m/2\rfloor, \ldots, \lfloor m/2\rfloor\}^d)$. The so-obtained random field $\{\overline{X}_i\}_{i\in\mathbb{Z}^d}$ is $(m+1)$-dependent, that is, for all $i, j \in \mathbb{Z}^d$, $\overline{X}_i$ and $\overline{X}_j$ are independent, if $\max_{q=1,\ldots,d} |i_q - j_q| > m+1$.

Lemma 1. *If $\Delta_p < \infty$ for some $p \geq 2$, then*

$$\lim_{m\to\infty} \sup_n \frac{\|S_n(X,b) - S_n(\overline{X},b)\|_p}{b_n} = 0\,.$$

Proof. By Proposition 3 in [16],

$$\|S_n(X,b) - S_n(\overline{X},b)\|_{\leq} \Big(2p \sum_{j\in\mathbb{Z}^d} b_{n,j}^2\Big)^{1/2} \Delta_p^{(m)}\,,$$

where $\Delta_p^{(m)}$ is the physical dependence measure for $X - \overline{X}$:

$$\Delta_p^{(m)} = \sum_{j\in\mathbb{Z}^d} \|(X_j - \overline{X}_j) - (X_j - \overline{X}_j)^*\|_p\,. \tag{13}$$

By Lemma 2 in [16], $\Delta_p < \infty$ implies that $\lim_{m\to\infty} \Delta_p^{(m)} = 0$. ☐

In the sequel, let $\overline{\Delta}_p = \sum_{j\in\mathbb{Z}^d} \|\overline{X}_j - \overline{X}_j^*\|_p$ denote the physical dependence measure of $\overline{X}$. Observe that $\Delta_p < \infty$ implies that $\overline{\Delta}_p = \sum_{j\in\{-m,m\}^d} \|\overline{X}_j - \overline{X}_j^*\|_p < \infty$, for all $m \in \mathbb{N}$.

(ii) Coefficient-averaging. This procedure was introduced by Peligrad and Utev [36] in the one-dimensional case. For each $l \in \mathbb{N}$, recall the definition of $c_{n,k}$ in (7). Set $\bar{b}_{n,j} \equiv \bar{b}_{n,j}(l) = \sum_{k\in\mathbb{Z}^d} c_{n,k} \mathbf{1}_{\{j\in I_k\}}$.

Lemma 2. *For each $m \in \mathbb{N}, l \in \mathbb{N}$, if $\overline{\Delta}_p < \infty$ for some $p \geq 2$, then*

$$\lim_{n\to\infty} \frac{\|S_n(\overline{X},b) - S_n(\overline{X},\bar{b})\|_p}{b_n} = 0\,.$$

Proof. Apply Theorem 1 and (8) to $\overline{X}$. ☐

(iii) Big/small blockings. Define, for $k \in \mathbb{Z}^d, l \in \mathbb{N}, l > m+1$,

$$\widetilde{I}_k \equiv \widetilde{I}_k(l) = \{i \in \mathbb{Z}^d : i_q \in \{lk_q+1, \ldots, lk_q+l-(m+1)\}, q = 1, \ldots, d\} \subset I_k\,.$$

Set $Y_k = \sum_{j\in\widetilde{I}_k} \overline{X}_k\,, k \in \mathbb{Z}^d$. Since $\{\overline{X}_i\}_{i\in\mathbb{Z}^d}$ are $(m+1)$-dependent, by the construction of $\{\widetilde{I}_k\}_{k\in\mathbb{Z}^d}$, $\{Y_k\}_{k\in\mathbb{Z}^d}$ are i.i.d. random variables. Consider $S_n(Y,c) = \sum_{j\in\mathbb{Z}^d} c_{n,j} Y_j$.

Lemma 3. *For each $m \in \mathbb{N}$, if $\overline{\Delta}_p < \infty$ for some $p \geq 2$ and* (9) *holds, then*

$$\lim_{l\to\infty}\lim_{n\to\infty}\frac{\|S_n(\overline{X},\overline{b}) - S_n(Y,c)\|_p}{b_n} = 0\,.$$

Proof. Observe that $S_n(\overline{X},\overline{b})-S_n(Y,c) = \sum_{k\in\mathbb{Z}^d} c_{n,k}\sum_{j\in I_k\setminus\widetilde{I}_k}\overline{X}_j$. By Theorem 1,

$$\begin{aligned}\|S_n(\overline{X},\overline{b}) - S_n(Y,c)\|_p &\leq \Big\{2p[l^d-(l-(m+1))^d]\sum_{k\in\mathbb{Z}^d}c_{n,k}^2\Big\}^{1/2}\overline{\Delta}_p\\ &= \Big\{(2p)\Big[1-\Big(\frac{l-(m+1)}{l}\Big)^d\Big](l^dc_n^2)\Big\}^{1/2}\overline{\Delta}_p\,.\end{aligned}$$

Note that (9) implies $\lim_{n\to\infty} l^dc_n^2/b_n^2 = 1$, whence the desired result follows. □

(iv) Triangular array of weighted i.i.d. random variables. Now we establish a central limit theorem for $S_n(Y,c)$. Recall that Y depends on $l, m \in \mathbb{N}$.

Lemma 4. *For each $m \in \mathbb{N}, l > m+1$, if $\overline{\Delta}_p < \infty$ for some $p \geq 2$ and* (10) *holds, then*

$$\frac{S_n(Y,c)}{l^{d/2}c_n} \Rightarrow \mathcal{N}(0,\sigma_{m,l}^2)$$

with

$$\sigma_{m,l}^2 = \sum_{i\in\{m+1-l,\ldots,l-m-1\}^d}\mathbb{E}(\overline{X}_0\overline{X}_i)\prod_{r=1}^d\Big(1-\frac{m+1+|i_k|}{l}\Big)\,.$$

Proof. It is equivalent to prove a central limit theorem for $\sum_{j\in\mathbb{Z}^d}c_{n,j}\overline{Y}_j/c_n$ with $\overline{Y}_j = Y_j/l^{d/2}$. By straight-forward calculation, $\mathbb{E}(\overline{Y}_1^2)/l^d = \sigma_{m,l}^2$. Then, (10) yields the desired result. □

Proof of Theorem 2. Combining Lemmas 1–4 yields that

$$\lim_{m\to\infty}\lim_{l\to\infty}\limsup_{n\to\infty}\frac{\|S_n(X,b) - S_n(Y,c)\|_2}{b_n} = 0\,.$$

By [16], proof of Theorem 3.1 therein, $\sigma^2 = \lim_{m\to\infty}\lim_{l\to\infty}\sigma_{m,l}^2$. The desired result follows. □

At last, we prove Corollary 1. By the product form (12), we can write

$$b_{n,j} = \prod_{q=1}^d b_{n,j_q}(q) \quad\text{and}\quad b_n = \prod_{q=1}^d b_n(q) \tag{14}$$

with $b_{n,j_q}(q) := \sum_{i=1}^n a_{i-j_q}(q)$ and $b_n(q) := [\sum_{j_q\in\mathbb{Z}} b_{n,j_q}^2(q)]^{1/2}$. Accordingly, for each $l, k \in \mathbb{Z}^d$, we write $c_{n,k_q}(q) := \sum_{j_q=k_q l+1}^{(k_q+1)l} b_{n,j_q}(q)$ and $c_n(q) := [\sum_{k_q\in\mathbb{Z}} c_{n,k_q}^2(q)]^{1/2}$.

In the sequel, for the sake of simplicity, for $j, k \in \mathbb{Z}^d$ write $b_{n,j}(q) \equiv b_{n,j_q}(q)$ and $c_{n,k}(q) \equiv c_{n,k_q}(q)$ and view them as functions of $j_q, k_q \in \mathbb{Z}$, respectively.

Proof of Corollary 1. Fix $l \in \mathbb{N}$ and recall that $c_{n,k}$ depends on l. We first show $\sup_{j\in\mathbb{Z}^d} |c_{n,j}|/c_n \to 0$ as $n \to \infty$. Write

$$\sup_{j\in\mathbb{Z}^d} \frac{|c_{n,j}|}{c_n} \leq \prod_{q=1}^d \sup_{j\in\mathbb{Z}^d} \frac{|c_{n,j}(q)|}{c_n(q)}, \tag{15}$$

and it converges to zero by the fact that $\sup_{j\in\mathbb{Z}^d} |c_{n,j}(q)|/c_n \to 0$ as $n \to \infty$. It suffices to show $\sup_{j\in\mathbb{Z}^d} |b_{n,j}(q)|/b_n(q) \to 0$ as $n \to \infty$ for all $q = 1, \ldots, d$, which was proved in Peligrad and Utev [35], p. 448.

Next, we show that the product form (12) implies that (8) and (9) hold. This result is an extension of Peligrad and Utev [36], Lemma A.1. We prove by induction. Note that now (9) becomes

$$\lim_{n\to\infty} \frac{1}{b_n^2} \sum_{k\in\mathbb{Z}^d} \sum_{j\in I_k} \Big| \prod_{q=1}^d b_{n,j}^2(q) - \prod_{q=1}^d c_{n,k}^2(q) \Big| = 0\,. \tag{16}$$

When $d = 1$, it was shown in [36] that (8) and (16) holds for $\sum_{k\in\mathbb{Z}} a_k^2 < \infty$.

Suppose (8) and (16) have been proved for $d-1$. We prove (8) for d and the proof of (16) is similar and omitted. By the inequality that for any real numbers $\alpha_q, \beta_q, q = 1, \ldots, d$,

$$\Big(\prod_{q=1}^d \alpha_q - \prod_{q=1}^d \beta_q\Big)^2 \leq 2\Big[\Big(\prod_{q=1}^{d-1} \alpha_q - \prod_{q=1}^{d-1} \beta_q\Big)^2 \alpha_d^2 + \Big(\prod_{q=1}^{d-1} \beta_q^2\Big)(\alpha_d - \beta_d)^2\Big],$$

we bound $\sum_k \sum_j [\prod_{q=1}^d b_{n,j}(q) - \prod_{q=1}^d c_{n,k}(q)]^2 \leq 2(\Phi_n^{(1)} + \Phi_n^{(2)})$ with

$$\Phi_n^{(1)} = \sum_{k_1,\ldots,k_{d-1}} \sum_{j_1,\ldots,j_{d-1}} \Big[\prod_{q=1}^{d-1} b_{n,j}(q) - \prod_{q=1}^{d-1} c_{n,k}(q)\Big]^2 \times \sum_{k_d} \sum_{j_d} b_{n,j}^2(d)$$

and

$$\begin{aligned} \Phi_n^{(2)} &= \sum_{k_1,\ldots,k_{d-1}} \sum_{j_1,\ldots,j_{d-1}} \prod_{q=1}^{d-1} c_{n,k}^2(q) \sum_{k_d} \sum_{j_d} \Big[b_{n,j}(d) - c_{n,k}(d)\Big]^2 \\ &= l^{d-1} \prod_{q=1}^{d-1} c_n^2(q) \sum_{k_d} \sum_{j_d} \Big[b_{n,j}(d) - c_{n,k}(d)\Big]^2. \end{aligned}$$

By induction,

$$\frac{\Phi_n^{(1)}}{b_n^2} = \frac{\sum_{k\in\mathbb{Z}^{d-1}}\sum_{j\in\mathbb{Z}^{d-1}}\left[\prod_{q=1}^{d-1} b_{n,j}(q) - \prod_{q=1}^{d-1} c_{n,k}(q)\right]^2}{\prod_{q=1}^{d-1} b_n^2(q)} = o(1)\,,$$

and similarly, $\Phi_n^{(2)}/b_n^2 \sim \sum_{k_d}\sum_{j_d}[b_{n,j}(d) - c_{n,k}(d)]^2/b_n^2(d) = o(1)$ by (8) with $d = 1$. We have thus obtained (8) for all $d \in \mathbb{N}$. □

4 An invariance principle

We consider weak convergence in the space $D[0,1]^d$ consisting of functions 'continuous from above with limits from below' (see Bickel and Wichura [2] for details). For $t \in [0,1]^d$, consider $S_n(t) \equiv \sum_{i_1=1}^{\lfloor nt_1\rfloor}\cdots\sum_{i_d=1}^{\lfloor nt_d\rfloor}\xi_i \in D[0,1]^d$. This time we have

$$S_n(t) = \sum_{j\in\mathbb{Z}^d} b_{nt,j}X_j \quad \text{with} \quad b_{nt,j} := \sum_{i_1=1}^{\lfloor nt_1\rfloor}\cdots\sum_{i_d=1}^{\lfloor nt_d\rfloor} a_{i-j}. \tag{17}$$

Theorem 3. *Suppose there exists $H \in (0,1)^d$ such that*

$$\lim_{n\to\infty}\frac{b_{\lfloor sn\rfloor}^2(q)}{b_n^2(q)} = s^{2H_q},\ \textit{for all } s \in [0,1], q = 1,\dots,d\,, \tag{18}$$

and there exists p such that

$$p \ge 2,\ p > \max_{q=1,\dots,d}\frac{1}{H_q} \quad \textit{and} \quad \Delta_p < \infty\,. \tag{19}$$

Then, $\{S_n(t)/b_n\}_{t\in[0,1]^d}$ converges weakly in $D[0,1]^d$ to the fractional Brownian sheet with Hurst index H.

Remark 3. When (18) holds for some $H \in (0,1)^d$ with $\max_{q=1,\dots,d} H_q^{-1} < 2$, condition (19) becomes $\Delta_2 < \infty$. Otherwise, we need to assume finite higher-than-second-order moment to establish the tightness. A similar phenomena was observed in the one-dimensional case ([9], Theorem 3.2).

Example 2. Due to the product structure, it suffices to provide examples of $\{a_{i_q}\}_{i_q\in\mathbb{Z}}$ such that (18) holds for each $q = 1,\dots,d$. Several examples have been provided in Dedecker et al. [9], Examples 1–4. We summarize them below.

(i) Fix $\alpha \in (0,1/2)$, and set $a_0 = 1, a_i = \Gamma(i+\alpha)/(\Gamma(\alpha)\Gamma(i+1))$ for $i \ge 1$. Then $H = \alpha + 1/2$.

(ii) Fix $\alpha \in (0,1/2)$, and set $a_i = (i+1)^{-\alpha} - i^{-\alpha}$ for $i \ge 1$. Then $H = 1/2 - \alpha$.

(iii) Fix $\alpha \in (1/2, 1)$ and set $a_i \sim i^{-\alpha} l(i)$ for $i \geq 1$ with any slowly varying function l at infinity. Then $H = 3/2 - \alpha$.

(iv) Fix $\alpha > 1/2$ and set $a_i \sim i^{-1/2}(\log i)^{-\alpha}$ for $i \geq 1$. Then $H = 1$.

Example 3 (Fractionally integrated random fields)**.** Case (i) in Example 2 above corresponds to the *fractionally integrated random fields*, generated by *back-shift operators*. Let B_q denote the back-shift operator on the q-th coordinate of the random fields $\{X_i\}_{i\in\mathbb{Z}^d}$: $B_q X_i = X_{i_1,\dots,i_{q-1},i_q-1,i_{q+1},\dots,i_d}$. Then, for $\alpha_q \in (0, 1/2)$,

$$(I - B_q)^{-\alpha_q} X_j := \sum_{i=0}^{\infty} a_i(q) X_{j_1,\dots,j_{q-1},j_q-i,j_{q+1},\dots,j_d}\,, j \in \mathbb{Z}^d\,,$$

with $a_i(q)$ defined in Example 2, (i). Thus, fractionally integrated random fields defined by

$$\xi_j = (I - B_1)^{-\alpha_1} \cdots (I - B_d)^{-\alpha_d} X_j\,, j \in \mathbb{Z}^d$$

fit in our model (1) with coefficients $a_i = \prod_{q=1}^d a_{i_q}(q), i \in \mathbb{Z}^d$, where $\{a_{i_q}(q)\}_{i\in\mathbb{Z}_+}$ corresponds to B_q as above and $a_{i_q}(q) = 0$ for $i_q < 0$. This generalizes the fractional autoregressive integrated moving average (FARIMA) processes (see e.g. [46] and references therein) to random fields ([27]). Note that Wu and Shao [46] also established an invariance principle for the so-called *Type II* fractional integrated processes, which are slightly different from our model (1).

Theorem 3 follows as usual from the convergence of finite-dimensional distributions and tightness (see e.g. Bickel and Wichura [2]), which are proved below separately.

Proposition 1 (Convergence of finite-dimensional distributions)**.** *Suppose $\Delta_2 < \infty$ and for some $H \in (0, 1)^d$ (18) holds. then the finite-dimensional distributions of $\{S_n(t)\}_{t\in[0,1]^d}$ converge to that of a fractional Brownian sheets $\{\mathbb{B}^H_t\}_{t\in[0,1]^d}$ with Hurst index H.*

Proof. We start with some new notations following (17). This time the product structure of the coefficients yields, for $n = (n(1), \dots, n(d)) \in \mathbb{Z}^d_+$,

$$b_{n,j} = \prod_{q=1}^d b_{n(q),j_q}(q) \quad \text{with} \quad b_{n(q),j_q}(q) = \sum_{i=1}^{n(q)} a_{i-j_q}(q)\,, j \in \mathbb{Z}^d\,. \tag{20}$$

Define accordingly $b_n(q) = [\sum_{j_q\in\mathbb{Z}} b^2_{n(q),j_q}(q)]^{1/2}$. Similarly define $c_{n,k}(q) = \sum_{j_q=k_q l+1}^{(k_q+1)l} b_{n(q),j_q}(q)/l$ and $c_{n,k} = \prod_{q=1}^d c_{n(q),k_q}(q)$. As before, for $n \in \mathbb{Z}^d_+$ and $j, k \in \mathbb{Z}^d$, write

$$b_{n,j}(q) \equiv b_{n(q),j_q}(q) \quad \text{and} \quad c_{n,k}(q) \equiv c_{n(q),k_q}(q)\,, \tag{21}$$

for the sake of simplicity.

Fix $m \in \mathbb{N}$. Take arbitrary $t^{(1)}, \ldots, t^{(m)} \in [0,1]^d$ and write $n_r = (\lfloor n t_1^{(r)} \rfloor, \ldots, \lfloor n t_d^{(r)} \rfloor) \in \mathbb{Z}_+^d$ for $r = 1, \ldots, m$. Then we can write

$$\sum_{r=1}^{m} \lambda_r S_n(t^{(r)}) = \sum_{j \in \mathbb{Z}^d} \Big(\sum_{r=1}^{m} \lambda_r b_{n_r,j} \Big) \xi_j \,.$$

By Theorem 2, it suffices to show that $\{\widetilde{b}_{n,j} = \sum_{r=1}^m \lambda_r b_{n_r,j}\}_{n,j}$ are regular and

$$\widetilde{b}_n^2 := \sum_{j \in \mathbb{Z}^d} \Big(\sum_{r=1}^{m} \lambda_r b_{n_r,j} \Big)^2 \sim b_n^2 \mathrm{Var}\Big[\sum_{r=1}^{m} \lambda_r \mathbb{B}^H(t^{(r)}) \Big] \,. \tag{22}$$

We first prove (22). Observe that

$$\widetilde{b}_n^2 = \sum_{j \in \mathbb{Z}^d} \sum_{r=1}^{m} \sum_{s=1}^{m} \lambda_r \lambda_s b_{n_r,j} b_{n_s,j} = \sum_{r=1}^{m} \sum_{s=1}^{m} \lambda_r \lambda_s \sum_{j \in \mathbb{Z}^d} b_{n_r,j} b_{n_s,j} \,.$$

For fixed r, s, recall that for $j \in \mathbb{Z}^d$, by our notation (21), $b_{n_r,j}(q)$ depends only on $j_q \in \mathbb{Z}$. Then,

$$\begin{aligned}
\sum_{j \in \mathbb{Z}^d} b_{n_r,j} b_{n_s,j} &= \sum_{j \in \mathbb{Z}^d} \prod_{q=1}^{d} b_{n_r,j}(q) b_{n_s,j}(q) \\
&= \sum_{j_1, \ldots, j_{d-1} \in \mathbb{Z}} \prod_{q=1}^{d-1} b_{n_r,j}(q) b_{n_s,j}(q) \sum_{j_d \in \mathbb{Z}} b_{n_r,j}(d) b_{n_s,j}(d) \\
&= \prod_{q=1}^{d} \sum_{j_q \in \mathbb{Z}} b_{n_r,j}(q) b_{n_s,j}(q) \\
&= \prod_{q=1}^{d} \sum_{j_q \in \mathbb{Z}} \frac{1}{2} \Big[b_{n_r,j}^2(q) + b_{n_s,j}^2(q) - (b_{n_r,j}(q) - b_{n_s,j}(q))^2 \Big] \\
&= \prod_{q=1}^{d} \frac{1}{2} \Big[b_{n_r}^2(q) + b_{n_s}^2(q) - b_{|n_r - n_s|}^2(q) \Big] \,.
\end{aligned}$$

We have thus shown that

$$\widetilde{b}_n^2 = \sum_{r=1}^{m} \sum_{s=1}^{m} \lambda_r \lambda_s \prod_{q=1}^{d} \frac{1}{2} \Big[b_{n_r}^2(q) + b_{n_s}^2(q) - b_{|n_r - n_s|}^2(q) \Big] \,.$$

On the other hand, by the covariance formula (2),

$$\operatorname{Var}\Big[\sum_{r=1}^{m}\lambda_r\mathbb{B}^H(t^{(r)})\Big]$$
$$=\sum_{r=1}^{m}\sum_{s=1}^{m}\lambda_r\lambda_s\prod_{q=1}^{d}\frac{1}{2}\Big[(t_q^{(r)})^{2H_q}+(t_q^{(s)})^{2H_q}-|t_q^{(r)}-t_q^{(s)}|^{2H_q}\Big].$$

Now (22) follows from (18) by recalling that $b_n^2=\prod_{q=1}^d b_n^2(q)$.

Next we check that $\{\widetilde{b}_{n,j}\}_{j\in\mathbb{Z}^d}$ are regular. Accordingly define $\widetilde{c}_{n,k}=\sum_{j\in I_k}\widetilde{b}_{n,j}/l^d$ and $\widetilde{c}_n=[\sum_{k\in\mathbb{Z}^d}\widetilde{c}^{\,2}_{n,k}]^{1/2}$. Observe that $\widetilde{c}_{n,k}=\sum_{r=1}^m\lambda_r c_{n_r,j}$. Then, conditions (8), and (9) become

$$\frac{1}{\widetilde{b}_n^2}\sum_{k\in\mathbb{Z}^d}\sum_{j\in I_k}(\widetilde{b}_{n,j}-\widetilde{c}_{n,k})^2\to 0 \text{ and } \frac{1}{\widetilde{b}_n^2}\sum_{k\in\mathbb{Z}^d}\sum_{j\in I_k}|\widetilde{b}^2_{n,j}-\widetilde{c}^{\,2}_{n,k}|\to 0 \text{ as } n\to\infty\,. \tag{23}$$

The first part follows from the observation that for each r, $(\widetilde{b}_{n,j}-\widetilde{c}_{n,k})^2\le m\sum_{r=1}^m\lambda_r^2(b_{n_r,j}-c_{n_r,k})^2$ and $\sum_{k\in\mathbb{Z}^d}\sum_{j\in I_k}(b_{n_r,j}-c_{n_r,k})^2/b_n^2\to 0$ as $n\to\infty$. To show the second part of (23), observe that

$$\sum_{k\in\mathbb{Z}^d}\sum_{j\in I_k}|\widetilde{b}^2_{n,j}-\widetilde{c}^{\,2}_{n,k}|\le\sum_{k\in\mathbb{Z}^d}\sum_{j\in I_k}|\widetilde{b}_{n,j}-\widetilde{c}_{n,k}||\widetilde{b}_{n,j}+\widetilde{c}_{n,k}|$$
$$\le\Big(\sum_{k\in\mathbb{Z}^d}\sum_{j\in I_k}|\widetilde{b}_{n,j}-\widetilde{c}_{n,k}|^2\Big)^{1/2}\Big(\sum_{k\in\mathbb{Z}^d}\sum_{j\in I_k}|\widetilde{b}_{n,j}+\widetilde{c}_{n,k}|^2\Big)^{1/2},$$

where the first term in the last product is of order $o(\widetilde{b}_n)$ (by the first part of (23)), while the second term $O(\widetilde{b}_n)$, whence (23) follows.

At last, condition (10) becomes $\lim_{n\to\infty}\sup_{j\in\mathbb{Z}^d}|\widetilde{c}_{n,j}|/\widetilde{c}_n=0$. To see this, observe that the second part of (23) implies that $l^d\widetilde{c}^{\,2}_n\sim\widetilde{b}^2_n$. It then follows from (22) and (9) that $\widetilde{c}^{\,2}_n\sim Cc_n^2$ for some constant $C>0$. The rest of the proof is similar to the control of (15) and omitted. We have proved the regularity of $\{\widetilde{b}_{n,j}\}_{n,j}$ and thus the proposition.

□

Proposition 2 (Tightness). *If there exists p such that* (19) *holds, then the process $\{S_n(t)/b_n\}_{t\in[0,1]^d}$ is tight in $D[0,1]^d$.*

Proof. We will apply Lavancier [26], Corollary 3. By slightly modifying the argument therein, it suffices to show that there exists constants $\beta>1,p>0,C>0$, such that for all $t\in(0,1)^d$ and n large enough,

$$\|S_n(t)\|_p^p\le Cb_n^p\prod_{q=1}^d t_q^{\beta}\,. \tag{24}$$

By Theorem 1, for $p \geq 2$,

$$\|S_n(t)\|_p = \Big\| \sum_{j\in\mathbb{Z}^d} b_{nt,j} X_j \Big\|_p \leq (2p)^{1/2} b_{nt} \Delta_p = (2p)^{1/2} \prod_{q=1}^d b_{nt}^{(q)} \Delta_p \,. \qquad (25)$$

Observe that (18) means that $b_n(q)$ is regularly varying. Now, by Taqqu [40], Lemma 4.1, for all $\gamma_q > 0$, there exists $C_q > 0$ such that $b_{nt}/b_n \leq C_q t^{H_q-\gamma_q}$, uniformly on $[0,1]$ for n larger than some n_q. Now, (25) can be controlled by, for n large enough and some constant C,

$$\|S_n(t)\|_p^p \leq C b_n^p \prod_{q=1}^d t^{p(H_q-\gamma_q)} \Delta_p \,.$$

If p satisfies (19), then one can choose $\gamma_q > 0$ small enough so that $p(H_q - \gamma_q) > 1$. It then follows that (24) holds with $\beta = \min_q p(H_q - \gamma_q) > 1$. We have thus proved the tightness. $\square$

5 Discussions

We compare our results with Surgailis [38] and Lavancier [28]. Surgailis proved more general results in the sense that he considered general functionals of linear random fields, with independent innovations. However, it assumes finite moments of any order of the innovations, which is not necessary in the special case of invariance principle for fractional Brownian sheets. It is an interesting problem that whether the arguments shown here can be extended to the more general settings of [38].

Next, we compare our results and Lavancier [28]. The latter assumed the following assumption on the stationary random fields:

H1 The random field $\{X_i\}_{i\in\mathbb{Z}^d}$ is *weakly stationary* (i.e., with shift-invariant covariance structure), centered and has a bounded spectral density f_X, and there exists a random field $\{\mathbb{B}_t\}_{t\in(0,\infty)^d}$ such that

$$\Big\{ \frac{1}{n^{d/2}} \sum_{i_1=1}^{\lfloor nt_1 \rfloor} \cdots \sum_{i_d=1}^{\lfloor nt_d \rfloor} X_i \Big\}_{t\in(0,\infty)^d} \overset{\text{f.d.d.}}{\to} \{\mathbb{B}_t\}_{t\in(0,\infty)^d} \,. \qquad (26)$$

By Theorem 1, (ii), $\Delta_2 < \infty$ implies that the spectral density of $\{X_i\}_{i\in\mathbb{Z}^d}$ is bounded (see e.g. Fan and Yao [17], Theorem 2.11 for the case $d = 1$). By [16], Proposition 4, $\Delta_2 < \infty$ also implies (26). Therefore, our assumption on the random fields is stronger than **H1**. Results in [28] are also more general in the sense that they do not assume the product structure of coefficients, and they cover various limiting objects including fractional Brownian sheets.

However, as an invariance principle for fractional Brownian sheets, our setting is more general than [28]. Indeed, the limiting objects in [28] are described through a linear mapping (from $L^2(\mathbb{R}^d)$ to $L^2(\Omega, \mathcal{B}, \mathbb{P})$), which is not explicit in general ([28], Remark 4). In particular, the fractional Brownian sheets can be interpreted from this characterization, *only* when $\{X_i\}_{i\in\mathbb{Z}^d}$ are i.i.d. and the coefficients are of product form.

We conclude by comparing our assumptions on the coefficients with the ones in [28], Theorem 5 (where strong white noise and product form of coefficients were assumed). Due to the nature of the spectral analysis approach therein, for each q, the coefficients $\{a_i(q)\}_{i\in\mathbb{Z}}$ were assumed to be the Fourier coefficients of certain function $\widehat{a}^{(q)} \in L^2([-\pi, \pi])$:

$$\widehat{a}^{(q)}(\omega) = \sum_{j\in\mathbb{Z}} a_j(q) \mathrm{e}^{-\sqrt{-1}j\omega}, q = 1, \ldots, d.$$

Focus on $\widehat{a}^{(q)}(\omega_q)$ and omit the index q from now on. In [28], Theorem 5 (see also Remark 6), it was assumed that

$$\widehat{a}(\omega) \sim C|\omega|^{-\alpha}, \text{ as } \omega \to 0, \text{ for some } \alpha \in (0, 1/2), C > 0\,. \tag{27}$$

By results on trigonometric series (see e.g. Zygmund [49], Chapter V, Theorems 2.6 and 2.24), (27) is equivalent to assume $a_j \sim C_1 j^{\alpha-1}$ as $j \to \infty$ for some constant $C_1 > 0$, which is a special case in Example 2, (iii). The other cases of Example 2 are not covered by (27). Therefore, our assumptions on the coefficient are more general than [28] in the case of invariance principles for factional Brownian sheets.

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